I BECAME A TEACHER OF HIGH SCHOOL MATHEMATICS with the goal of ensuring success for all my students, particularly those from groups that were traditionally excluded from advanced mathematics coursework. Ironically, eight years into what began as a journey to bring mathematical learning to a broader audience of students, I find that I am the one who has learned a lot—about mathematics and about what it means to “teach for understanding.” In particular, I have learned how my own thinking about, and understanding of, mathematics affect the ways that I structure learning opportunities for my students that can either shut down or open avenues for success.

MATHEMATICAL NON-SENSE

As a beginning teacher, I did my best to explain algebra clearly to my students. I was full of idealism and confidence, and I was ready to share my knowledge of the subject. Just as my teachers had done, I used an overhead projector and colorful pens to illustrate the steps for solving problems—linear equations, proportions, points of intersection, and so on. I used clever (if not entirely realistic) examples and stories to help students remember different procedures. And of course, I assigned a sufficient number of practice problems to allow students to gain facility with various skills. The work that I received from students was not what I had hoped. Granted, some students did their work just as I had shown and obtained correct answers. However, many did what appeared to be mathematical procedures but obtained results that made no sense. And a few students did not even know where to begin (or did not care). I began to realize that this job was not going to be easy.

Discuss:

• How do your experiences as a learner of mathematics affect the ways in which you have come to think about teaching and learning mathematics?

Since I was not willing to accept that a significant portion of my students were unable to succeed, I tried to detect what made learning algebra so difficult for them. Listening to the comments of my students (some of whom had no qualms about being brutally honest), I came to understand that much of the frustration that students experienced in learning mathematics arose because they did not see

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connections with—or within—the work that they were doing. From their perspective, they saw no rhyme or reason for learning what appeared to be discrete pieces of impersonal mathematical knowledge that did not fit together in ways that made sense.

I decided that I would try to provide experiences that would enable students to apply the procedures that we were studying, thereby creating chances for personal interactions with “the mathematics.” I remembered a session that I had attended at a local conference for teachers of mathematics. The presenter shared an activity in which he asked students to measure the length (not including the eraser) and the weight of various pencils. The sharpened pencils were the same brand, but their lengths varied. The students plotted the data on a coordinate grid and drew a line of best fit. The presenter then challenged students to make predictions about pencils that they had not measured, determine a mathematical relationship between pencil length and weight, and calculate an estimate of the weight of the eraser.

When I tried this activity with my algebra students—who had previously studied linear equations—they were noticeably more engaged in the task. In working through the investigation, most were able to attach meaning to the slope and y-intercept values, concepts that had previously seemed so alien. My students began to develop deeper understandings of the connections among such ideas as slope, rate of change, linearity, and y-intercept, as evidenced by the way in which they approached problems related to the investigation. Their strategies and justifications, as well as their results, made sense.

An interesting extension of this activity might be to give each group a different brand of pencil and compare any variation in the resulting slopes. What would the results indicate about the different brands?

Discuss:

• How do the ways in which your students experience mathematics learning affect their personal understandings of mathematics?

SEARCHING FOR PERSONAL MEANING

Although the pencil activity certainly did not entirely clear up the students’ confusion, some of the pieces began to come together for them. This success started me thinking about designing additional engaging learning experiences that would allow students to make sense of mathematical concepts. I then realized something unnerving—the shallowness of my own conceptual understanding of mathematics. It was one of the most profound moments for me as a teacher. I realized that despite having studied mathematics throughout seventeen years of schooling, I lacked a deep personal understanding of mathematical concepts. The questions and responses generated by the pencil experiment stretched, extended, and refined my own thinking about linear relationships. For me, too, some of the pieces of mathematical knowledge that I had accumulated started to come together in new and exciting ways.

Although I had always done well in mathematics courses, rarely was I encouraged to discuss my understanding of an idea with classmates, given opportunities to tie mathematics to personal experiences, or asked explicitly to make connections among concepts in ways that held meaning for me. My learning of mathematics was confined to internalizing objective subject matter—page-by-page and lecture-by-lecture—detached from personal experience and understanding. Success was defined by passing tests full of problems that mirrored homework exercises, tests that required that I use formulas and techniques but that seemed to reflect little interest in whether I could explain or understand how and why the methods worked. Mathematics was to me a separate reality—a game in which I manipulated values and variables in an effort to obtain the correct answer. This mathematics was the subject that I learned so well and that so many others came to loathe. The toughest question to face was the one that changed my outlook on my role as a teacher: Was I creating the same sort of impersonal experiences with mathematics for my own students?

Deciding that I wanted my students to experience mathematics differently meant that I needed to shift my entire way of thinking about the subject. In trying to open my mind to new methods of teaching, I had to confront the fact that my own understanding of mathematics consisted of procedures, formulas, and rules that were not necessarily organized into any coherent whole. How could I as a teacher provide my students with substantive and different mathematical learning experiences? Furthermore, how might these experiences affect the way in which students come to think about and understand mathematics? And how would I determine whether students had been successful? This question-posing led me to realize that I needed to learn not only about teaching mathematics for understanding but also to work toward understanding mathematics for teaching.

Discuss:

• What opportunities do you provide for students to engage in meaningful interactions with and around important concepts of mathematics, that is, opportunities that allow them to explore and enrich their understandings of these concepts?
The summer following my first year of teaching, I attended the Monterey Bay Area Mathematics Project in Santa Cruz, California. Rather than present participants—all teachers of mathematics—with a "bag of tricks" to use in the classroom, (that being the title of a professional development session I once attended at which we were shown a multitude of gimmicks to use in our teaching) the instructors expected us to be learners of mathematics. Instead of throwing pieces of mathematical knowledge at us, they provided opportunities for us to explore and form connections among mathematical ideas. It was fascinating (and sometimes frustrating) to discover that problems that seemed simple on the surface became more and more complex as we were challenged to re-form our prior thinking. The carefully planned interactions—mathematical and interpersonal—and thoughtful questioning from the project's instructors served as an example of a pedagogy aimed at empowering students to work actively toward making sense of mathematics. An entirely new perspective on mathematical teaching and learning began to open up to me.

During the three-week-long project, we spent a lot of time as teacher-students thinking about and discussing the mathematical representation of relationships between two variables. The materials that were used came primarily from The Language of Functions and Graphs (Shell Centre for Mathematical Education 1985). The idea was for us to revisit the mathematics of functions while broadening our pedagogical lens. Rather than begin with presentations of formal definitions and rules, lessons were designed to move from intuitive notions to more formal mathematical constructions.

One of the early sessions that week involved a problem from a section of The Language of Functions and Graphs entitled “Are Graphs Just Pictures?” It asked us to sketch a graph of the speed of a golf ball over time. In the problem, we were given the fact that the ball followed an unlikely trajectory—from the club’s head along a perfect parabola straight into the hole. Although this exercise may sound trite, the discussions that ensued were anything but. As we thought about the general shape of such a graph, we also discussed such questions as, What happens to the speed just after the ball is hit? How do acceleration and deceleration appear on the graph? What might be the limits of the range of the graph? How is the graph related to the distance that the ball has traveled? Although the point of the problem, which was designed for students who were beginning their studies of functions, was to realize how and why the shape of the speed-versus-time graph is dissimilar to the ball’s trajectory, our exploration as teacher-students led us to think about many more mathematical concepts and connections. Once again, I sensed that the pieces of my mathematical past were coming together.

While we continued engaging in these sorts of mathematically rich interactions, I realized that with a deeper understanding of the connections among concepts, the stagnant mathematical procedures that I learned long ago became both more meaningful and more flexible. As a teacher, I started to appreciate the value of allowing students opportunities to exchange and challenge ideas while they grapple with making sense of mathematical concepts.

Discuss:

- How has your understanding of mathematics changed as a result of meaningful interactions with learners of mathematics?

**RE-FORMING MY TEACHING OF MATHEMATICS**

As Brookfield (1995) indicates,

> We may espouse philosophies of teaching that we have learned from formal study, but the most significant and most deeply embedded influences that operate on us are the images, models, and conceptions of teaching derived from our own experiences as learners. (Brookfield 1995, p. 49)

Following this experience of mathematical exploration and sense-making, I believed that I had gained insight into the intent of the NCTM’s Standards. I had theoretically known what was meant by the phrase “teaching for understanding”—I wanted students to understand what I was teaching well enough to get the right answers on their tests! However, my summer experience challenged me to form more complex views while I began to realize that teaching for understanding means working toward providing opportunities and support so that students can engage in meaningful thinking with and about mathematical concepts.

When I started thinking about shifting my classroom practice toward having students build personal understandings of concepts, I decided that including an occasional application activity was not enough. The summer experience had shown me that problems did not need to be connected with real life. More important to consider were the ways in which students experience the mathematics itself as meaningful. If they have opportunities to become personally involved in constructing their understanding of mathematics—whether through the hands-on use of manipulative models, collecting and analyzing numerical data, exploring abstract patterns, or explaining key concepts—students will be more likely to think of mathematics
as something that makes sense and to think of themselves as able to be successful learners of mathematics.

Although I was not entirely certain about how to accomplish such a pedagogical transition, I knew that I could no longer play the role of “giver of knowledge.” Fortunately, just before the start of the next school year, I heard from one of my sum-

temporary colleagues about College Preparatory Mathematics: Change from Within (CPM), a set of Standards-based textbooks designed for interactive, meaning-centered learning. I was able to convince the administration—with whose support I had attended the project—to allow me to use the CPM algebra materials for my classes. The experience was incredible. Lessons no longer centered on what I was doing and saying from behind the overhead projector. The ideas that students shared and conversations among students became the foci for learning. The units were structured to enable students to move from familiar to abstract, from exploration to explanation, and from seeing pieces of the picture to connecting pieces to form beautiful and complex images. My role became twofold: I was an informed facilitator of students’ interactions, thereby ensuring through thoughtful planning and strategic questioning that progress was being made toward understanding key mathematical concepts and ideas; and I took part as a teacher-participant in the action of learning, along with students. Being so involved in the learning processes meant that I had a far greater awareness of my students’ progress than I previously had. I therefore understood when it was appropriate to review concepts, demonstrate procedures, or give targeted practice, and I was able to do so in ways that helped students make sense of the mathematics.

The insights that I have experienced during seven years of actively listening to students think about and interact with meaningful mathematics have often challenged and amazed me. My understanding of mathematics has been stretched in ways that I could have never imagined back when I thought that I knew it all. Most satisfying, however, has been to witness my students showing interest in, and appreciation for, what they do because they are able to make sense of mathematics. In addition, my students have become much more articulate in identifying areas of confusion. All these changes have translated into significant improvements in their mathematical achievement, as measured not only by standardized tests of skills and procedures but also with more meaning-centered assessments of conceptual understanding, such as students’ mathematical essays that explain the concept of linear relationships by using words, symbols, and graphs.

Discuss:

- How might schools encourage and enable colleagues to work collaboratively to develop and implement meaningful learning activities for their students?

A CHALLENGE TO THE TRADITIONAL MODEL OF TEACHING

The rationale for presenting knowledge as separate from one’s self and ignoring personal understanding has been questioned by Wendy Atwell-Vasey, who asserts that the traditional practices of education have led to a situation in which learning is driven by . . . mechanisms that rely on external motivations and standards, not on the most powerful of learning motivations, the learner’s felt desire to be better connected to the world and to secure more power to function in it. (Atwell-Vasey 1998, p. 48)

I argue that NCTM Standards-based reforms in mathematics are an attempt to get at “the learner’s felt desire” to understand the world in which he or she lives. An approach to teaching that empowers students to take control of their learning provides a powerful incentive to students to do their best. If the goal is to guide students to understand concepts and make connections among important ideas, mathematics cannot be (re)presented as a collection of eternal truths that have come down from above to the mortals below (on whom much of its meaning will be lost). Figure 1 shows a representation of that model. Students must be allowed to construct mathematical meaning from carefully planned

In this model, content knowledge is acquired through repeated practice of procedures and memorization of relationships. A teacher must pass on such knowledge to students. Personal understanding of mathematics is not an explicit component of this model; it lies outside the model and is not directly addressed. Once a teacher becomes an “expert” (that is, completes advanced mathematics coursework and training), her or his own learning is typically considered to be complete.

The stagnant mathematical procedures I learned long ago became both more meaningful and more flexible
experiences and thoughtful interactions that give them opportunities to connect and broaden previous understandings while learning new procedures and concepts. Mathematical meaning is not something that I as a teacher can simply “give” to students. Rather, it is something that I can help them achieve by allowing students to see mathematics as a product of human experience, thought, and interaction, as shown in the model in figure 2, as well as by allowing them to see that mathematics makes sense.

**CONCLUDING THOUGHTS**

I am led to question how mathematics education in the United States might be reshaped into something resembling the vision expressed in NCTM’s Principles and Standards for School Mathematics (NCTM 2000). Assuming that this vision is one with which people agree, what would be needed for such ideas about mathematics teaching and learning to become more than just rhetoric? It is a tall order, indeed, to ask those of us who learned to view mathematics as an impersonal collection of skills and procedures to radically shift this perception and teach mathematics as a set of interconnected concepts whose structure students should understand through sense-making experiences. Although the field of study remains basically the same, the level of insight and understanding required to be successful becomes much richer. Indeed, in Knowing and Teaching Elementary Mathematics, Liping Ma argues that teachers need a “profound understanding of fundamental mathematics” (Ma 1999, p. xxiv).

To become a more effective (and reflective) teacher, I have needed more than just reading or hearing about different pedagogical strategies. Most important have been opportunities to re-form the pieces of my mathematical past and experience firsthand what personally understanding a concept means. It will always be a “work in progress,” in the sense that I am continually refining and re-forming my mathematical understanding as a result of new interactions. Although I am confident that I can plan for and support meaningful learning for students of mathematics, I am also comfortable knowing that I will always have a lot to learn from them.

**REFERENCES**


